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AALBORG UNIVERSITY
DENMARK

Papers, Volume 1 1962-1985

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Publication date:
2006

Document Version
Publisher's PDF, also known as Version of record

[Link to publication from Aalborg University](#)

Citation for published version (APA):
Thoft-Christensen, P. (2006). *Papers, Volume 1 1962-1985*. Department of Civil Engineering, Aalborg University.

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CHAPTER 7

SOME EXPERIENCE FROM APPLICATION OF OPTIMIZATION TECHNIQUE IN STRUCTURAL RELIABILITY¹

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Summary

Determination of the so-called reliability index β used in the second order reliability theory for structures is formulated as a constrained optimization problem. It is suggested to use the constrained Fletcher-Powell technique in solving the optimization problem.

Some experience from using this technique on a simple problem is presented. Especially convergence and the choice of starting points are treated. Finally it is shown for the same problem that the reliability index β is rather sensitive with regard to the standard deviations for the basic variables.

1. INTRODUCTION

As far back as the late 1920's work on reliability analysis of structures based on probability theory can be found. But serious progress in this field has mainly taken place in the last decade or so. Several different ways of attacking important reliability problems in structural engineering have now been developed. Although important unsolved problems still exist it is now obvious that these theories have reached such a degree of development that it is safe to use them in practical design work.

Methods of reliability analysis of structures are usually classified on the basis of the types of approximation made. The most advanced and complex theories are the so-called *level III* theories. They can be characterized as being exact probabilistic methods of analysis based on knowledge of the full distribution of all relevant variables. In opposition to these exact theories idealizations are made in the *level II* theories. Idealizations can e.g. be that the statistics of the basic variables can be described in terms of means and standard deviations. Finally, in *level I* theories the appropriate

¹ Byggningsstatistiske Meddelelser, Vol. 48, No.2, 1977, pp. 31-44.

levels of reliability are provided by specification of a number of partial safety factors to be used in designing the single structural element.

In this paper only level II theories are considered. They have in the last few years mainly been used in deducing partial safety factors in connection with a level I theory. It can be expected that future international codes will be based on partial safety factors derived by such considerations in a rational way. But the level II theories have also a more direct area of application in relation to more complicated structures such as offshore structures where a more expensive analysis can be accepted.

The purpose of the present paper is not to contribute to the theoretical development of new level II theories but to present some experiences from numerical determination of the so-called reliability index β on the basis of a well-known optimization technique. It has always been recognized that calculation of this reliability index can be formulated as an optimization problem but to the author's knowledge no concrete data have been presented so far. It might be of some interest to observe that such a formulation is extremely simple and that it is a usable alternative to other iterative procedures.

In sections 2 and 3 a simplified second-moment reliability theory and an advanced optimization technique, respectively, are presented. A very simple example is used in section 4 to illustrate the use of optimization methods in calculation of the reliability β . Finally some results regarding the influence of the standard deviations of the variables on the index β are discussed in chapter 5.

2. RELIABILITY INDEX β

In this section a very brief representation of the second order reliability index β is given in its simplest form. A more satisfactory and detailed description of this method to evaluate the safety of structures can be found in several papers, e.g. Cornell [1], Ditlevsen [2] and Hasofer & Lind [3]. In these papers and in several other papers the limitations of the theory and several important refinements are analysed in details. Only the basic theory will be used here as it clearly illustrates the most important aspects in evaluation of the index β by optimization methods.

In this relatively new way of evaluating the safety of a structure first of all the relevant basic random variables X_1, X_2, \dots, X_n must be selected. The basic variables can be geometrical quantities (e.g. the area of the cross-section of a beam), material strength (e.g. rupture or yield strength) and external loads (e.g. traffic loads, wave or wind loads). Each of the random variables $X_i, i = 1, 2, \dots, n$ are in the present second-moment theory assumed to be sufficiently determined by the mean values $E[X_i]$ and the variances $\text{Var}[X_i]$. For the sake of simplicity the basic variables $X_i, i = 1, 2, \dots, n$ are assumed to be uncorrelated. Further the variables $X_i, i = 1, 2, \dots, n$ are assumed to have no lower or upper limits, i.e. the values of X_i belongs to the interval $[-\infty; \infty]$. If the latter assumption cannot be made for a basic variable, some kind of transformation might be necessary. It has been suggested by Ditlevsen & Skov [7] to use a standardized transformation, namely a logarithmic transformation to avoid ambiguity in the definition of the index β .

When the basic variables $X_i, i = 1, 2, \dots, n$ are chosen and their mean values $E[X_i]$ and standard deviations $D[X_i] = (\text{Var}[X_i])^{1/2}$ are determined the next step is to normalize (or reduce) all the variables in the following way

$$Y_i = \frac{X_i - E[X_i]}{D[X_i]} \quad , \quad i = 1, \dots, n \quad (1)$$

Then

$$E[Y_i] = 0 \text{ and } D[Y_i] = 1 \quad (2)$$

It is now assumed that the limit state of the structure can be determined by an equation of the form

$$f(Y_1, Y_2, \dots, Y_n) = 0 \quad (3)$$

in such a way that positive values of f indicate safe sets of normalized basic variables (the safe domain) and negative values of f indicate unsafe sets of variables (the domain of failure). The limit state equation (3) can in this connection indicate e.g. an actual failure situation, but it can also e.g. indicate a critical deflection. In figure 1 the limit state function f is shown in a two-dimensional case. Note that the origin O of the normalized (or reduced) coordinate system is placed in the safe domain.

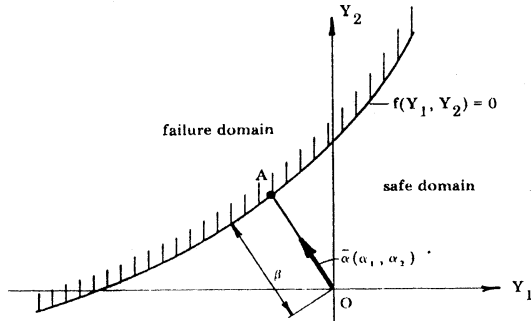


Figure 1.

The second order reliability index β is defined as the shortest distance from the origin to the limit state curve in the normalized coordinate system (see figure 1). This definition is a reasonable generalization of the situation for the one-dimensional case. Clearly, it can be expected that a large value of β will correspond to a little probability of failure although it is not possible in general to connect β with an exact probability of failure. If the basic variables X_i , $i = 1, 2, \dots$

, n are normally distributed and if the limit state function f is linear in the basic variables it can be shown that the probability of failure is

$$P_f = \Phi(-\beta) \quad (4)$$

where Φ is the normal distribution function.

It is reasonable to expect that equation (4) can be used to give an approximate value for the probability of failure P_f if the limit state curve is almost linear in the neighbourhood of the design point A (see figure 1) and if the distributions for the basic variables do not deviate too much from the normal distribution. It is outside the scope of this paper to discuss this important problem more detailed. The list of references at the end of this paper can be referred to.

Assuming the limit state function f to be differential it is easy to see that the distance β and the unit vector $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ given by $\overrightarrow{OA} = \beta \bar{\alpha}$, where A is the design point, can be determined by solving the following $n + 1$ equations

$$\left. \begin{aligned} \alpha_i &= - \left[\sum_{k=1}^n \left(\frac{\partial f}{\partial Y_k} \right)^2 \right]^{-\frac{1}{2}} \frac{\partial f}{\partial Y_i} \quad , \quad i = 1, \dots, n \\ f(\beta \alpha_1, \dots, \beta \alpha_n) &= 0 \end{aligned} \right\} \quad (5)$$

In the two-dimensional case illustrated in figure 1 the equations (5) are

$$\begin{aligned}\alpha_1 &= - \left[\left(\frac{\partial f}{\partial Y_1} \right)^2 + \left(\frac{\partial f}{\partial Y_2} \right)^2 \right]^{-\frac{1}{2}} \frac{\partial f}{\partial Y_1} \\ \alpha_2 &= - \left[\left(\frac{\partial f}{\partial Y_1} \right)^2 + \left(\frac{\partial f}{\partial Y_2} \right)^2 \right]^{-\frac{1}{2}} \frac{\partial f}{\partial Y_2} \\ f(\beta\alpha_1, \beta\alpha_2) &= 0\end{aligned}\tag{6}$$

The equations (5) (and (6)) are very suitable for solving by an iterative method.

In this paper the reliability index is determined by formulating the problem as an optimization problem in the following simple way. The distance b from a point $P(y_1, \dots, y_n)$ on the limit surface f to the origin O is given by

$$b^2 = \sum_{i=1}^n Y_i^2\tag{7}$$

The square of the reliability index β^2 is therefore the minimum value of b^2 subjected to the constraint

$$f(Y_1, \dots, Y_n) = 0\tag{8}$$

So the constrained optimization problem to be solved is the following

$$\begin{aligned}\beta^2 &= \min \left[\sum_{i=1}^n Y_i^2 \right] \\ \text{subject to } f(Y_1, Y_2, \dots, Y_n) &= 0\end{aligned}\tag{9}$$

3. OPTIMIZATION METHOD

In this paper the constrained optimization problem (9) is solved by the so-called *constrained Fletcher-Powell technique*, which will be briefly described in this section. It will only be used here in relation to the problem (9) where only one constraint is present. However, the constrained Fletcher-Powell technique is a much more general method as it can be used on nonlinear objective functions $F(X_1, \dots, X_n)$ subject to any number of equality constraints

$$G_k(X_1, \dots, X_n) = 0, \quad k = 1, 2, \dots, m\tag{10}$$

The functions F and G_k are assumed continuously differentiable. A detailed description of the iterative procedure used in this paper is given by Haarhoff & Buys [4]. The so-called CONMIN-algorithm is used in the numerical calculations presented in the sections 4 and 5 in a form presented by Kuester & Mize [5]. A CDC Cyber 72 computer belonging to Aalborg University Centre has been used for all computer runs.

The above-mentioned constrained optimization problem with m constraints (10) is formulated as an unconstrained problem with the new objective function ϕ , where

$$\phi = F - \sum_{k=1}^m \lambda_k G_k + B \sum_{k=1}^m G_k^2\tag{11}$$

B is a large positive constant (10 - 30) and $\lambda_k = 1, 2, \dots, m$ are constants which are updated for each new iteration.

The unconstrained problem with the objective function ϕ is solved by the effective *unconstrained Fletcher-Powell technique*. It is an iterative search technique, which can be briefly described by the following steps:

- (1) Choose a starting point.
- (2) Compute a direction of search. Initially the direction of steepest descent is used.
- (3) Conduct a one-dimensional search in the direction chosen in step (2).
- (4) Make a convergence check. If convergence is achieved, the procedure is terminated. If convergence is not achieved a new search direction is chosen at the minimum point from the last search. The new search direction is a modified gradient direction, and λ_k is updated in a certain way to improve convergence.
- (5) A new one-dimensional search is performed in the new direction and the process is repeated until convergence is obtained. Convergence is achieved when $G_k = 0$, $k = 1, 2, \dots, m$ and $F = \phi$.

It is recommended to use several alternative-starting points if the function ϕ is suspected to be multimodal. In all computer runs described in chapters 4 and 5 exactly the same system parameters are used as in the standard form of the programme presented by Kuester & Mize [5].

4. A NUMERICAL EXAMPLE

To illustrate the application of the above-mentioned optimization technique a very simple example is chosen from a Danish textbook [6] on structural reliability theory. In figure 2 a reinforced concrete beam of rectangular cross-section in bending is shown.

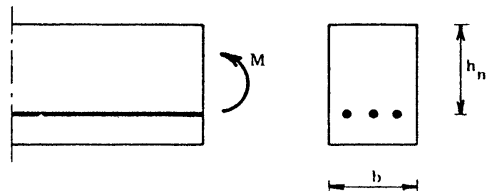


Figure 2.

The yield moment M_F in bending is given by the equation

$$M_F = \eta F_j \sigma_F h_n \quad (12)$$

where the coefficient η is determined by

$$\eta = 1 - k \frac{F_j \sigma_F}{b h_n \sigma_c} \quad (13)$$

The meaning of the symbols in equations (12) and (13) is given in table 1, where also some values for the means $E[X]$ and standard deviations $D[X]$ for the variables are shown. These statistical values are the same as in the example in the textbook [6].

Symbol	Basic variable	$E[X]$	$D[X]$
$X_1 = M$	Bending moment	0.01 MNm	0.003 MNm
$X_2 = h_n$	Effective height	0.3 m	0.015 m
$X_3 = \sigma_F$	Yield stress	360 MPa	36 MPa
$X_4 = F_j$	Area of reinforcement	$2.26 \times 10^{-4} \text{ m}^3$	$0.113 \times 10^{-4} \text{ m}^3$
$X_5 = k$	Form factor	0.5	0.05
$X_6 = b$	Width	0.12 m	0.006 m
$X_7 = \sigma_c$	Compressive strength of concrete	40 MPa	6 MPa

Table 1.

The limit state of this structure is determined by the following equation

$$M_F - M = 0 \quad (14)$$

or with the symbols defined in table 1

$$(1 - X_5 \frac{X_4 X_3}{X_6 X_2 X_7}) X_4 X_3 X_2 - X_1 = 0 \quad (15)$$

In the book [6] it is argued that the coefficient of variation, $V[\eta]$, for

$$\eta = 1 - X_5 \frac{X_4 X_3}{X_6 X_2 X_7} \quad (16)$$

is very small compared with the coefficients of variation for X_4 , X_3 , and X_2 so that η can be considered a constant, namely

$$\eta = 1 - 0.5 \frac{2.26 \times 10^{-4} \times 360}{0.12 \times 0.30 \times 40} = 0.972 \quad (17)$$

The number of basic variables can in this case be reduced from seven to four and the limit state equation (15) is

$$0.972 \times X_2 X_3 X_4 - X_1 = 0 \quad (18)$$

The next step in determining the reliability index β is to introduce the normalized (reduced) basic variables

$$Y_i = \frac{X_i - E[X_i]}{D[X_i]} \quad , \quad i = 1, \dots, 7 \quad (19)$$

in the equations (15) and (18).

The reliability index β can then be calculated as described in section 2 by iteration as it is done in the book [6] for the simplified case (18) with the result $\beta = 3.40$. It turns out that due to symmetry the number of variables can be further reduced from four to three during the iteration process.

By using the optimization technique presented in chapter 3 exactly the same result $\beta = 3.40$ is obtained and the design point A is also the same, namely

$$(Y_1, \dots, Y_4) = (2.59, -0.80, -1.89, -0.80) \quad (20)$$

Based on equation (15) one gets $\beta = 3.41$ and thereby the validity of reducing the number of variables from seven to four is confirmed. The design point is now

$$(Y_1, \dots, Y_7) = (2.62, -0.83, -1.87, -0.79, 0.04, -0.02, -0.01) \quad (21)$$

showing that the first four variables are only changed a little and that the three last ones are very close to 0.00 in accordance with the above conclusion.

In both cases (equations (15) and (18)) the optimization method has been used with several starting points. Except when the choice of starting point is close to the origin O of the reduced system of coordinates the computing time is only slightly influenced by the choice of starting point. The computing time was for the case with four variables approximately 1.0 sec. and with seven variables approximately 3.7 sec.

No special precautions were taken to decrease the computing time. The system data concerning convergence were used unaltered from the standard programme.

From this numerical example and from a few other examples where the optimization technique has been used it seems to be advantageous to use + 1 or -1 as starting values for the basic variables. The value + 1 can be recommended when the basic variable is a »loading variable» whereas -1 is a good starting value for »strength variables» and »geometrical variables». In the two situations above good starting points are therefore

$$(Y_1, Y_2, Y_3, Y_4) = (1, -1, -1, -1) \quad (22)$$

and

$$(Y_1, \dots, Y_7) = (1, -1, -1, -1, 1, -1, -1) \quad (23)$$

respectiv $\left[\left(\frac{\partial f}{\partial Y_1} \right)^2 + \left(\frac{\partial f}{\partial Y_2} \right)^2 \right]^{-\frac{1}{2}}$ ely.

Although the computing time is lower, of course, when the problem is simplified by reducing the number of basic variables from seven to four, access to an optimization programme as used here makes such a simplification unnecessary. In return for using a little more computer time the time spent estimating whether the variation of one or more basic variables is unimportant can be saved.

Number of iterations	β	Computer time CP sec.
2	4.36	0.17
5	4.19	0.31
10	3.58	0.53
20	3.44	1.02
50	3.41	2.36
82	3.41	3.72

Table 2.

The effectivity of the unconstrained Fletcher-Powell technique in its standard form is illustrated in table 2. The same structure with the above data and with seven basic variables is used in this check of convergence. As starting point the point (23) is used.

It is seen from these figures that the method in this case has been very satisfactory. With only 10 iterations (0.53 sec.) the β -value is only 5% from the exact value 3.41.

5. SENSITIVITY OF THE β -METHOD

As mentioned earlier in this paper level II theories besides being used in deducing partial safety factors have a more direct area of application, namely in analysis of the safety of structures. In such an application of the second-moment reliability theory presented above the mean values $E[X_i]$ and standard deviations $D[X_i]$ for the basic variables have to be evaluated and also information of the probability density functions is necessary. Sometimes it is possible to decide on the basis of tests and statistical considerations whether the density functions can be assumed to be Log normal, normal,

extreme, etc. with reasonable accuracy. With regard to $E[X_i]$ and $D[X_i]$ these numbers are usually determined by tests or by experience from similar structures, materials, load cases, etc.

In this section the sensitivity of the β -method with respect to variation of the standard deviation $D[X]$ is considered. One might argue that it is unfortunate if the reliability index β is too sensitive. A small decrease of $D[X]$ would then give a great increase of β and therefore also a decrease of the probability of failure P_F . Naturally, such an effect is not an objection to the theory itself, it only emphasizes that the value of $D[X]$ in such a case must be evaluated with considerable accuracy. However, it might in some cases be a great problem when using the β -method to get sufficiently good information of the value of $D[X]$. One might even be tempted to use too low values and thereby get a too low value for the probability of failure.

To get some information of the importance of having good evaluation of the standard deviations rather extensive numerical calculations have been carried out for the same simple example as in section 4. The simplified limit state equation (18) with only four variables has been used in this investigation but the results have been checked on the basis of the seven variables equation (15) with only small deviations. In the interpretation of the results below it is important to remember that the second-moment reliability theory is used here in a simplified form where for example no transformations are used on the basic variables X_i to change their intervals of definition.

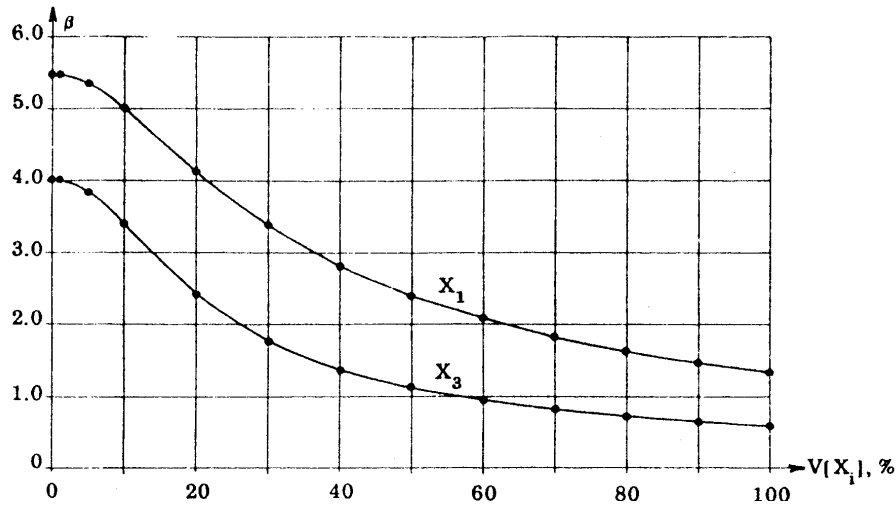


Figure 3

In equation (18) the basic variable X_1 is a »loading variable» and X_2 , X_3 , and X_4 are »strength variables» or »geometrical variables». In the investigation below a basic variable from each of these two groups of variables is chosen, namely X_1 and X_3 , i.e. the bending moment M and the yield stress σ_F respectively. Only the standard deviations $D[X_1]$ and $D[X_3]$ are therefore changed. $E[X_i]$, $i = 1, \dots, 4$ and $D[X_2]$ and $D[X_4]$ are unaltered from the values shown in table 1. β -values are shown in figure 3, where the coefficients of variation

$$V[X_1] = D[X_1]/E[X_1] \quad (24)$$

and

$$V[X_3] = D[X_3]/E[X_3] \quad (25)$$

vary from 0% to 100%. Note that a decrease of $V[X_1]$ from 30% as used in section 4 to 20% gives an increase of β from 3.40 to 4.14 equivalent to a decrease of the probability of failure from 3×10^{-4} to 8×10^{-5} if the equation (4) can be used as an approximation. Likewise, an increase of $V[X_3]$ from 10% to 20% gives a decrease of β from 3.40 to 2.41 equivalent to an increase of the probability of failure from 4×10^{-4} to 8×10^{-3} .

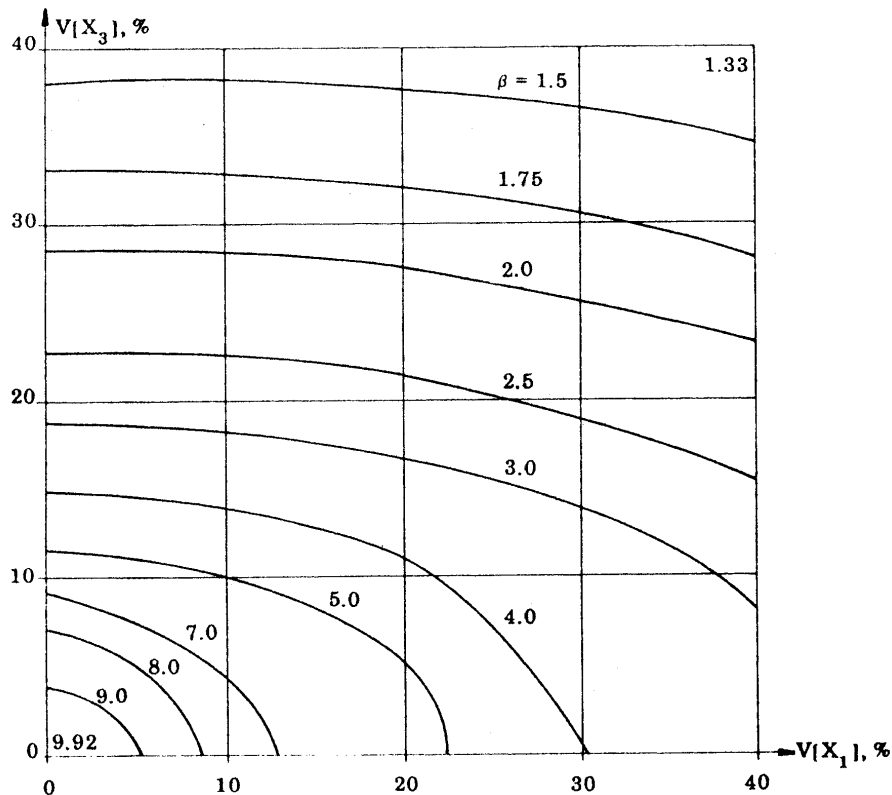


Figure 4.

Additional information of the sensitivity of the β -method is given in figure 4, where β -values can be read for all combinations of $V[X_1]$ and $V[X_3]$ in the interval [0% ; 40%]. The minimum value for β is 1.33 and the maximum is 9.92. It is seen that a change from $(V[X_1], V[X_3]) = (30\%, 10\%)$ to e.g. $(10\%, 5\%)$ gives an increase of β from 3.40 to 6.87 equivalent to a decrease of the probability of failure from 3×10^{-4} to a value less than 10^{-11} . Note also that for values of $V[X_3]$ in the neighbourhood of 30% the sensitivity with respect to $V[X_1]$ is relatively small, but for smaller values of $V[X_3]$ the β -value is strongly affected by a change of $V[X_1]$.

It is important to realize that this strong sensitivity with regard to changes in the standard deviations of the variables is not an undesirable effect. On the contrary, the idea behind the β -method is to use the standard deviations as measures for the reliability. The main point to be stated on basis of the above considerations is therefore that unreliable values for the standard deviations result in an unreliable value for the reliability index β .

6. CONCLUSIONS

The only tenable conclusion, which can be drawn with respect to application of optimization techniques in structural reliability analysis, is that for the particular example shown the method has been successful. One can ask whether evaluation of the reliability index by an optimization technique is more advantageous than by an iteration technique. It has not been the purpose of this paper to try to answer that question, but one can at least say that optimization techniques are simple tools, which should not be considered rivals but rather alternative methods to solve the same problem. One might argue that a main advantage of using an iterative method in solving the equations (6) is that it can be used without a computer although its application is then confined to a rather small number of basic variables. The optimization method used in the investigation is only an effective method if a computer is at hand, but then it seems to be unnecessary to try to reduce the number of variables.

It must be remembered that only analysis and not design has been treated in this paper. Further work is necessary before it can be decided whether the same optimization method is appropriate in solving design problems and also to be used in connection with deducing partial safety factors.

With regard to the sensitivity of the β -method investigated in section 5 the experience is again confined to a particular example. Therefore only temporary conclusion can be drawn now. However, it seems to be rather obvious that when using the β -method in evaluation of the safety of a structure it is absolutely necessary to get reliable values for the standard deviations $D[X_i]$. It is apparently not sufficient just to choose values within a limit of e.g. 10%. The required accuracy cannot be concluded on the basis of the limited amount of data presented here.

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